First-Passage Processes in Financial Markets

Jun-ichi Inoue[a], **Hikaru Hino**[a], **Naoya Sazuka**[b] and **Enrico Scalas**[c]

[a]Graduate School of Information Science and Technology, Hokkaido University
N14-W9, Kita-ku, Sapporo 064-0814, Japan
j_inoue@complex.eng.hokudai.ac.jp, hino@chaos1.complex.eng.hokudai.ac.jp

[b]System Technologies Laboratories, Sony Corporation
5-1-12 Kitashinagawa Shinagawa-ku, Tokyo 141-0001, Japan
Naoya.Sazuka@jp.sony.com

[c]Dipartimento di Scienze e Technologie, Universita del Piemonte Orientale
Via Bellini, 25 g I-15100 Alessandria, Italy
enrico.scalas@mfn.unipmn.it

Recently, lots of on-line trading services on the internet were constructed by several major banks such as the Sony Bank. They use a trading system in which foreign currency exchange rates change according to a first-passage process (FPP). Automatic FOREX trading systems are popular in Japan where many investors use a scheme called *carry trade* by borrowing money in a currency with low interest rate and lending it in a currency offering higher interest rates. In this paper, we investigate the statistical properties of the FPP from three different points of view, namely, *i) probabilistic-theoretical analysis*, *ii) empirical data analysis of BTP futures* (BTP is the middle and long term Italian Government bonds with fixed interest rates), and *iii) minority game modeling of the market*.

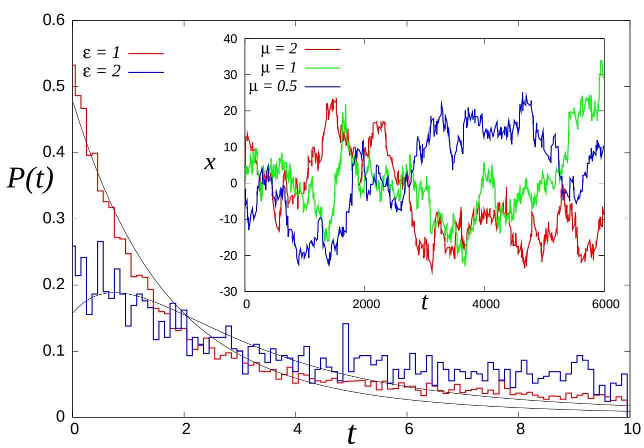

Figure 1: Theoretical prediction of the FPT pdf $P(t)$ (black solid-lines). The inset shows typical behaviour of a mixture of the NCPPs.

In our FPP, the stochastic variable is updated if and only if the time k-dependent stochastic variable x_k underlying the FPP holds $|x_{k+t} - x_k| \geq \epsilon$, where the duration t of the stochastic process.

i) probabilistic-theoretical analysis: We assume that the t is generated by the joint probability density $p(x,t) = \int d\boldsymbol{w}\, g(\boldsymbol{w}) p(x,t|\boldsymbol{w})$, $\int d\boldsymbol{w}\,(\cdots) \equiv \int_0^\infty d\mu \int_{-\infty}^\infty d\theta \int_0^\infty d\sigma\,(\cdots)$. $p(x,t|\boldsymbol{w})$ denotes the one-point probability density for a single normal compound Poisson process (NCPP) starting at the position $x=0$ for $t=0$ is given by $p(x,t|\boldsymbol{w}) = \mathrm{e}^{-\mu t} \sum_{n=0}^\infty (\mu t)^n \mathrm{e}^{-(x-n\theta)^2/2n\sigma^2}/n!\sqrt{2\pi n \sigma^2}$. We should notice that the average price x of a single NCPP diverges as $\langle x \rangle \equiv \int_0^\infty dt \int_{-\infty}^\infty p(x,t|\boldsymbol{w}) dx = (\theta/\mu) \sum_{n=0}^\infty n$ for a constant θ, whereas it remains finite by taking into account the distribution of θ. In this paper, we assume that the parameters μ, θ and σ are all independent variables: $g(\boldsymbol{w}) = g(\mu,\theta,\sigma) = f_\mu(\mu) f_\theta(\theta) f_\sigma(\sigma)$ and choose each form of the functions f_μ, etc. reasonably. Then, it should be noted that the parameter σ is not a relevant parameter in the sense that one can naturally rescale the parameters such as $\theta/\sigma \to \theta, \epsilon/\sigma \to \epsilon$ to cancel it. For the f_μ, we assume $f_\mu(\mu) = (1/\beta) \sum_{l=\mu_1}^{\mu_\beta} \delta(\mu - l)$. This choice is justified from the fact that the inverse of relaxation time μ takes the value dependent on the time of the day (*Time-of-the-Day Dependence*). For instance, the μ might take 1/10 during the period $k \in [0,6]$ and takes 1/20 for $k \in [6,12]$, 1/30 for $k \in [12,18]$ and 1/10 for $k \in [18,24]$.

Thus, in general, the distribution of μ converges to the above form $f_\mu(\mu)$ in the long-observation limit $k \to \infty$ of the market price x. When we assume that the θ obeys a normal Gaussian $\mathcal{N}(0,1)$, we have the first-passage time (FPT) probability distribution function analytically as $P(t) = \beta^{-1}\sum_{l=\mu_1}^{\mu_\beta} \mathrm{e}^{-lt}\sum_{n=0}^{\infty}\left(\mu - \frac{n}{t}\right)\frac{(lt)^n}{n!}\int_{-\infty}^{\infty} D\theta H_n(\theta,\epsilon)$ where $D\theta \equiv d\theta \mathrm{e}^{-\theta^2/2}/\sqrt{2\pi}$ and we defined $H_n(\theta,\epsilon)$ as the difference between two error functions: $H(\sqrt{n}\theta - \epsilon/\sqrt{n}) - H(\sqrt{n}\theta + \epsilon/\sqrt{n})$. In Figure 1, the results for several widths ϵ of the rate-window are shown. We find that the analytical results of $P(t)$ agree with computer simulations. For $\epsilon < \sqrt{\pi/2}$, we have the mixture of exponentials with effective relaxation time Δ^{-1} as an approximation:

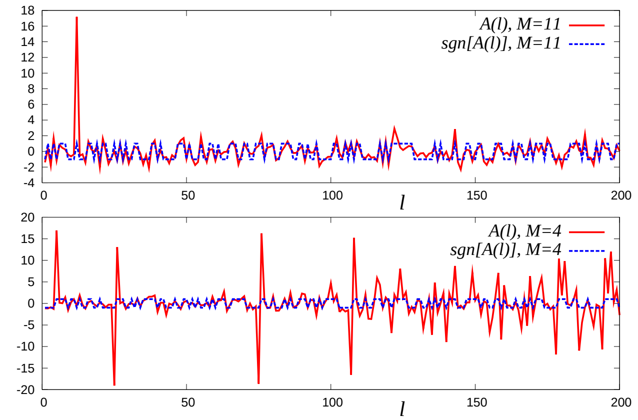

Figure 2: Typical behaviour of the total bit $A(l)$ where l is regarded as 'time step'. M denotes the market history length available to the traders.

$$P(t) \simeq \frac{1}{\beta}\left(1 - \epsilon\sqrt{\frac{2}{\pi}}\right)\sum_{l=\mu_1}^{\mu_\beta} l\mathrm{e}^{-lt} + \epsilon\Delta\sqrt{\frac{2}{\pi}}\mathrm{e}^{-\Delta t}, \quad \Delta \equiv -t^{-1}\log \mathrm{e}^{-lt}\sum_{n=0}^{\infty}\frac{(lt)^n}{n!\sqrt{n(n+1)}} > 0$$

From this expression, we find that in the $\epsilon \to 0$ limit, the mixture of exponential distributions that describes the point process for the market rate x underlying behind the FPP is recovered. For this FPT distribution, we evaluate the relevant statistics, namely, the residual life time w [4] and the Gini index [5]. In this paper, we calculate the w numerically and discuss the condition on which the so-called inspection paradox: $w > \int_0^\infty dt tP(t)$ takes place. **ii) Empirical data analysis:** To investigate to what extent the FPP changes the stochastic properties of financial markets, we investigate the Kullback-Leibler (KL) distance between the distribution of duration for the BTP future (from 20/01/1997 to 04/06/1997 with the skewness 0.087 and the kurtosis 23.75 for the return distribution) and the distribution of the first-passage time generated from the rate window with a width a. We calculate the KL distance K as a function of a. We find that the K monotonically increases as a increases. For instance, $K = 0.405643$ with $\langle t \rangle = 117.07, w = 63645.62$ for $a = 1$ and $K = 18.97$ with $\langle t \rangle = 19253.09, w = 71680.57$ for $a = 6$. This result tells us that the statistical properties of the FPP for the BTP future might change from the underlying stochastic process, namely, the BTP future itself. **iii) Minority game modeling:** To model the underlying stochastic process behind the FPP, the minority game might be one of the possible candidates. In our computer simulations, At each round l of the game, each trader i $(i = 1, \cdots, N)$ decides his (or her) decision: $b_i(l) = +1$ (*buy*) or $b_i(l) = -1$ (*sell*) to joint the minority group. Then, we evaluate the total bit $A(l) = (1/\sqrt{N})\sum_{i=1}^{N} b_i(l)$ for each round l. Then, the $A(l)$ follows complicated stochastic process as shown in Figure 2 and one might consider the FPP by regarding the total bit ('price') $A(l)$ as the underlying stochastic process. We evaluate the first passage time distribution and discuss the stochastic properties. The result will be reported in the conference.

[1] E. Scalas, *Chaos, Soliton & Fractals* **34**, pp. 33-40 (2007) [2] M. Montero and J. Masoliver, *European Journal of Physics B* **57**, pp.181-185 (2007) [3] http://moneykit.net [4] J. Inoue and N. Sazuka, to appear in *Quantitative Finance*, physics/0606040 [5] N. Sazuka and J. Inoue , *Physica A* **383**, pp.49-53 (2007) [5] N. Sazuka, J. Inoue and E. Scalas, submitted to *Physica A* (2008) [6] A.C.C. Coolen, *The Mathematical Theory Of Minority Games: Statistical Mechanics Of Interacting Agents*, Oxford University Press (2005) [7] D. Challet, M. Marsili and Yi.-C. Zhang, *Minority Games*, Oxford University Press (2005)